

# Chiral Fermion and Boundary State Formulation: Resonant Point-Contact Tunneling

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## Astract

We study a model of resonant point-contact tunneling of a single Luttinger-liquid lead, using the boundary state formulation. The model is described by a single chiral Fermion in one dimensional space with one point contact interaction at the origin. It is the simplest model of this type. By folding the infinite line we map the model onto a non-chiral model on the half infinite line and transcribe the point-contact tunneling interaction into a boundary interaction. The tunneling interaction yields a non-local effective action at the boundary. We explicitly construct the boundary state and evaluate the current correlation functions. The contact interaction is shown to act on the lower frequency modes more strongly.

## 1 INTRODUCTION

The one dimensional system is an exciting arena where techniques and ideas from different fields influence each other. The one-dimensional model describing the point-contact tunneling [1] between Luttinger-liquid [2, 3] leads certainly belongs to this category. We discuss the model in its simplest form, which has only a single chiral Fermion and a single lead. Despite its simple structure, the model exhibits its intriguing and essential features of models of this type, including the Kondo model [4, 5, 6, 7]. The model is defined on an infinite line with a point contact interactions at the origin. This description of the model is called “unfolded setup”. The models can be also studied in the “folded setup” [8, 9, 10, 11], which is more suitable for field theoretical analysis. The chiral fields on the infinite line are mapped onto non-chiral fields on a half infinite line and the mid point contact interactions

are transcribed into boundary interactions. The folded setup alludes that the boundary state formulation of string theory [12, 13, 14, 15, 16, 17, 18, 19] may be the most efficient framework to study the quantum impurity models of similar type. It is a simple matter to transcribe the models in the unfolded set up into the folded set up for the case of chiral boson models. However, it is not straightforward to transcribe the chiral Fermion models in the unfolded setup into the corresponding models in the folded setup.

In this paper, we discuss details of the folding procedure to transcribe the chiral Fermion model into the unfolded setup. In the folded setup, the model is described by a Dirac spinor with two components. Diving the infinite line into two half infinite lines, we get two boundaries. It is required that the boundary contact interaction term should be also extended appropriately. The Fermionic degrees of freedom of the impurity are represented by two-component spinors in the folded setup. Since the action for the spinors describing the impurity are only quadratic in Fermion fields, they can be integrated out. As a result, we obtain a non-local effective action for the Fermi fields of the Luttinger liquid on the boundary, which is the world line of the contact point. In order to apply the boundary state formulation of string theory, we interchange  $\tau$  and  $\sigma$ . It brings us the closed string picture of the model. We construct the exact boundary state, which reproduces the continuity condition at the contact point in the unfolded setup. Using the boundary state, we calculate the correlation functions of the current operators. It is found that the contact tunneling interaction acts on the lower frequency modes more strongly. The high frequency modes are affected only weakly by the contact interaction.

## 2 Model: Chiral Fermion Model

The one-dimensional model for the point-contact tunneling through a single Luttinger-liquid lead may be described by the following action of a single chiral Fermion field  $\eta$  on an infinite line

$$S = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma \int d\tau \left\{ \eta^\dagger (\partial_\tau + i\partial_\sigma) \eta + \delta(\sigma) d^\dagger \partial_\tau d + it\sqrt{2}\delta(\sigma) (\eta^\dagger d + \eta d^\dagger) \right\}. \quad (1)$$

This model has been briefly discussed in the literature [1]. But the field theoretical analysis of the model has never been completed. The Fermi fields  $d$  and  $d^\dagger$  depict the degrees of freedom of impurity located at the origin. They are the creation and annihilation operators of charge on the resonant state. In the action Eq.(1) we introduce a delta function in front of the kinetic term for the Fermi fields  $d$  and  $d^\dagger$  so that the action for those Fermi fields is defined only on the world line of the contact point. If the delta function were not present, we would encounter a product of two delta functions when we evaluate the effective boundary action by integrating out  $d$  and  $d^\dagger$ . It yields a divergence, which requires the renormalization of the coupling constant  $t$ . Here we avoid this unnecessary complication by simply defining the entire action for  $d$  and  $d^\dagger$  only on the worldline of the contact point. The action can be given also in terms of a chiral boson field. But for the purpose of field theoretical calculation the Fermion action is more preferable, since the interaction term is only quadratic in Fermi fields.

## 2.1 Chiral Fermion model in the Unfolded Setup

We may rewrite the action Eq.(1) in the following form by dividing it into the bulk action and the boundary action

$$S = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma \int d\tau \left\{ \eta^\dagger (\partial_\tau + i\partial_\sigma) \eta \right\} + \frac{1}{2\pi} \int d\tau \left\{ d^\dagger \partial_\tau d + it\sqrt{2} (\eta^\dagger d + \eta d^\dagger) \right\} \Big|_{\sigma=0}. \quad (2)$$

The action is defined on a two dimensional Euclidean space and all Fermi fields are anti-periodic in  $\tau$ . Thus, we are working on the theory at finite temperature. For the sake of simplicity we also scale some physical variables appropriately in such a way that the range of  $\tau$  is  $[0, 2\pi]$ . The equations of motion, which follow from the action Eq.(2) are given by

$$(\partial_\tau + i\partial_\sigma)\eta + it\sqrt{2}\delta(\sigma)d = 0, \quad \partial_\tau d - it\sqrt{2}\eta \Big|_{\sigma=0} = 0. \quad (3)$$

Since the Fermi fields are anti-periodic in  $\tau$  we may expand the Fermi fields in terms of normal modes

$$\eta(\tau, \sigma) = \sum_n \eta_n(\sigma) e^{in\tau}, \quad d(\tau) = \sum_n d_n e^{in\tau}, \quad (4a)$$

$$\eta^\dagger(\tau, \sigma) = \sum_n \eta_n(\sigma) e^{-in\tau}, \quad d^\dagger(\tau) = \sum_n d_n e^{-in\tau} \quad n \in \mathbf{Z} + 1/2. \quad (4b)$$

If we rewrite the equations of motion in terms of normal modes,

$$n\eta_n = -\partial_\sigma \eta_n - t\sqrt{2}\delta(\sigma)d_n \quad (5a)$$

$$nd_n = t\sqrt{2}\eta_n(0) = \frac{t}{\sqrt{2}} [\eta_n(0+) + \eta_n(0-)]. \quad (5b)$$

For Eq.(5a) to be defined continuously across the origin, the following condition must be imposed

$$-[\eta_n(0+) - \eta_n(0-)] - t\sqrt{2}d_n = 0. \quad (6)$$

Then it follows from Eqs.(5b,6),

$$\eta_n(0+) = \left( \frac{1 - \frac{t^2}{n}}{1 + \frac{t^2}{n}} \right) \eta_n(0-). \quad (7)$$

This condition should be understood as the continuity condition at the contact point, which as we shall see shortly, corresponds to the boundary condition in the folded setup.

## 2.2 Chiral Fermion Model in the Folded Setup

The folding procedure begins with splitting the infinite one dimensional space into two half infinite lines. We rename the Fermi fields on the second half line as  $\zeta$  and  $c$  to distinguish them from the Fermi fields  $\eta$  and  $d$  on the first half line

$$\begin{aligned} S = & \frac{1}{2\pi} \int_0^\infty d\sigma \int d\tau [\eta^\dagger (\partial_\tau + i\partial_\sigma) \eta + \delta(\sigma) d^\dagger \partial_\tau d] \\ & + \frac{1}{2\pi} \int_{-\infty}^0 d\sigma \int d\tau [\zeta^\dagger (\partial_\tau + i\partial_\sigma) \zeta + \delta(\sigma) c^\dagger \partial_\tau c] \\ & + \frac{it\sqrt{2}}{2\pi} \int_{\sigma=0} d\tau (\eta^\dagger d + \eta d^\dagger). \end{aligned} \quad (8)$$

For the time being we adopt the same boundary action as in the unfolded setup. The differential operator in the action,  $\partial$  should be understood as  $\frac{1}{2}(\vec{\partial} - \overleftarrow{\partial})$  [20]. Thus action for the Fermi field on the second half line may be written also as

$$\frac{1}{2\pi} \int_{-\infty}^0 d\sigma \int d\tau [\zeta^\dagger (\partial_\tau + i\partial_\sigma) \zeta] = \frac{1}{2\pi} \int_{-\infty}^0 d\sigma \int d\tau [\zeta (\partial_\tau + i\partial_\sigma) \zeta^\dagger]. \quad (9)$$

Splitting the infinite line into two half infinite lines, we create two boundaries. So appropriate boundary condition should be imposed on the boundaries of the two half infinite lines. It follows from Eq.(9) that we can choose one of the following two conditions

$$\eta(\tau, 0) = \zeta(\tau, 0)e^{i\theta}, \quad \eta^\dagger(\tau, 0) = \zeta^\dagger(\tau, 0)e^{-i\theta} \quad (10)$$

or

$$\eta(\tau, 0) = \zeta^\dagger(\tau, 0)e^{i\theta}, \quad \eta^\dagger(\tau, 0) = \zeta(\tau, 0)e^{-i\theta}, \quad (11)$$

as a boundary condition at  $\sigma = 0$ . Here  $\theta$  is a constant phase to be fixed later. We choose the second one Eq.(11) as the boundary condition for a reason to be clear shortly.

Then we take the parity transformation  $\sigma \rightarrow -\sigma$  on the second half line to merge two bulk actions into one. Exchanging the world sheet coordinates  $\tau \rightarrow \sigma$  and  $\sigma \rightarrow \tau$  we get a closed string picture of the model

$$\begin{aligned} S = & \frac{1}{2\pi} \int_0^\infty d\tau \int d\sigma [i\eta^\dagger (\partial_\tau - i\partial_\sigma) \eta + \delta(\sigma) d^\dagger \partial_\sigma d] \\ & + \frac{1}{2\pi} \int_0^\infty d\tau \int d\sigma [-i\zeta^\dagger (\partial_\tau + i\partial_\sigma) \zeta + \delta(\sigma) c^\dagger \partial_\sigma c] \\ & + \frac{it\sqrt{2}}{2\pi} \int_{\tau=0} d\sigma (\eta^\dagger d + \eta d^\dagger). \end{aligned} \quad (12)$$

At the same time we shift the phases of the Fermi fields as

$$\eta^\dagger \rightarrow -i\eta^\dagger, \quad \zeta^\dagger \rightarrow i\zeta^\dagger, \quad d^\dagger \rightarrow -id^\dagger, \quad c^\dagger \rightarrow ic^\dagger, \quad (13)$$

to cast the bulk action into the familiar action of the Dirac spinor with two components

$$S = \frac{1}{2\pi} \int_0^\infty d\tau \int d\sigma [\eta^\dagger (\partial_\tau - i\partial_\sigma) \eta + \zeta^\dagger (\partial_\tau + i\partial_\sigma) \zeta] \\ + \frac{1}{2\pi} \int_{\tau=0} d\sigma \left[ -id^\dagger \partial_\sigma d + ic^\dagger \partial_\sigma c + t\sqrt{2} (\eta^\dagger d + \eta d^\dagger) \right].$$

The path integral measure  $D[\eta, \eta^\dagger, \zeta, \zeta^\dagger]$  for the Fermi fields is invariant under this phase shift. From the closed string action Eq.(12) it is clear that  $\eta$  is the right mover (antiholomorphic function) and  $\zeta$  is the left mover (holomorphic function)

$$\zeta(\tau, \sigma) = \psi_L(\tau, \sigma), \quad \zeta^\dagger(\tau, \sigma) = \psi_L^\dagger(\tau, \sigma), \quad (14a)$$

$$\eta(\tau, \sigma) = \psi_R(\tau, \sigma), \quad \eta^\dagger(\tau, \sigma) = \psi_R^\dagger(\tau, \sigma). \quad (14b)$$

The boundary condition Eq.(11) is now read as

$$\eta(\tau, 0) = i\zeta^\dagger(\tau, 0)e^{i\theta}, \quad \eta^\dagger(\tau, 0) = i\zeta(\tau, 0)e^{-i\theta}. \quad (15)$$

If we choose  $e^{i\theta} = e^{-i\theta} = -1$ , this condition Eq.(15) can be indentified as the Neumann condition

$$\psi_L(\tau, 0) = i\psi_R^\dagger(\tau, 0), \quad \psi_L^\dagger(\tau, 0) = i\psi_R(\tau, 0) \quad (16)$$

(Note that we may represent the Fermi field operators  $\psi_L$  and  $\psi_R$  in terms of the boson fields  $\phi_L$  and  $\phi_R$  as

$$\psi_L = e^{-\frac{\pi}{2}i(p_L+p_R)} e^{-\sqrt{2}i\phi_L}, \quad \psi_R = e^{-\frac{\pi}{2}i(p_L+p_R)} e^{\sqrt{2}i\phi_R}. \quad (17)$$

In this representation the Neumann condition for the boson fields,  $\phi_L = \phi_R$  can be transcribed into Eq.(16) for the Fermi fields  $\psi$ .)

Now the bulk action can be written succinctly in terms of two component Dirac spinor  $\psi$

$$S[\psi] = \frac{1}{2\pi} \int_0^\infty d\tau \int d\sigma \bar{\psi} \gamma \cdot \partial \psi. \quad (18)$$

where  $\gamma^0 = \sigma_1$ ,  $\gamma^1 = \sigma_2$ ,  $\gamma^5 = \sigma_3 = -i\gamma^0\gamma^1$  and

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \quad (19)$$

As aforementioned the boundary interaction term should be also extended to be consistent with the folded setup

$$t\sqrt{2} (\eta^\dagger d + \eta d^\dagger) \rightarrow t (\eta^\dagger d + \eta d^\dagger + c^\dagger \zeta + c \zeta^\dagger) = it (\bar{\psi} \gamma^1 \xi - \bar{\xi} \gamma^1 \psi) \quad (20)$$

where we define the two-component spinor  $\xi$ , representing the degrees of freedom of the impurity as

$$\xi = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}. \quad (21)$$

Under the Neumann condition, which may be extended to include the Fermi fields  $\xi$

$$\psi_L = i\psi_R^\dagger, \quad \psi_L^\dagger = i\psi_R, \quad (22a)$$

$$\xi_L = i\xi_R^\dagger, \quad \xi_L^\dagger = i\xi_R, \quad (22b)$$

the two interaction terms of the boundary action Eq.(20) are equivalent to each other. The coupling constant  $t$  is scaled also in such a way that the folded model produces the same boundary condition as the unfolded model. Completing the folding procedure brings us the following action in the folded setup, to which the boundary state formulation is readily applicable

$$S = \frac{1}{2\pi} \int_0^\infty d\tau \int d\sigma \bar{\psi} \gamma \cdot \partial \psi + \frac{1}{2\pi} \int_{\tau=0} d\sigma \{ \bar{\xi} \gamma^1 \partial_\sigma \xi + it (\bar{\psi} \gamma^1 \xi - \bar{\xi} \gamma^1 \psi) \}. \quad (23)$$

### 3 Boundary State

The advantages of the action in the folded setup are evident: Since model is depicted by the free closed string bulk action and the boundary action, which produces only boundary (initial) condition at the initial time  $\tau = 0$ , the free closed string Hamiltonian can be employed to evaluate the correlation functions of physical operators. We are ready to apply the boundary state formulation, which may be the most efficient method to calculate the partition function and the correlation functions of various physical operators.

We note that the action is only quadratic in the Fermi field  $\xi$ . Integrating out  $\xi$  yields a boundary effective action for  $\psi$ . It is convenient to make use of the mode expansions on a cylinder

$$\psi(\tau, \sigma) = \sum_n \psi_n(\tau) e^{in\sigma}, \quad \bar{\psi}(\tau, \sigma) = \sum_n \bar{\psi}_n(\tau) e^{in\sigma}, \quad (24a)$$

$$\xi(\sigma) = \sum_n \xi_n e^{in\sigma}, \quad \bar{\xi}(\sigma) = \sum_n \bar{\xi}_n e^{in\sigma} \quad (24b)$$

to explicitly evaluate the effective boundary action. For the Fermi fields on a cylinder, two conditions are available; the anti-periodic condition

$$\psi(2\pi) = -\psi(0), \quad \bar{\psi}(2\pi) = -\bar{\psi}(0), \quad \xi(2\pi) = -\xi(0), \quad \bar{\xi}(2\pi) = -\bar{\xi}(0), \quad (25)$$

and the periodic condition

$$\psi(2\pi) = \psi(0), \quad \bar{\psi}(2\pi) = \bar{\psi}(0), \quad \xi(2\pi) = \xi(0), \quad \bar{\xi}(2\pi) = \bar{\xi}(0). \quad (26)$$

The former is called the Neveu-Schwarz (NS) sector where the normal modes are labeled by half-odd-integers,  $n \in \mathbf{Z} + 1/2$  and the latter is called the Ramond (R) sector where the normal modes are labeled by integers,  $n \in \mathbf{Z}$ . Here we choose the NS sector only, because the vacuum in the R sector carries a non-vanishing momentum while the vacuum in the NS sector does not. The physical vacuum belongs to the NS sector.

### 3.1 Effective Boundary Action

If we rewrite the action for the Fermi field  $\xi$  in terms of normal modes Eq.(24a,24b)

$$S[\bar{\xi}, \xi] = \frac{1}{2\pi} \int_{\tau=0} d\sigma \left\{ \bar{\xi} \gamma^1 \partial_\sigma \xi + it (\bar{\psi} \gamma^1 \xi - \bar{\xi} \gamma^1 \psi) \right\}, \quad (27)$$

we have

$$S[\bar{\xi}, \xi] = i \int_0^\infty d\tau \sum_n \left\{ n \bar{\xi}_{-n} \gamma^1 \xi_n + t (\bar{\psi}_{-n} \gamma^1 \xi_n - \bar{\xi}_{-n} \gamma^1 \psi_n) \right\}. \quad (28)$$

If we integrate out the Fermi field  $\xi$ , we obtain the effective action for  $\psi$

$$\int D[\bar{\xi}, \xi] \exp \{-S[\bar{\xi}, \xi]\} = \exp \left\{ -it^2 \sum_n \frac{1}{n} \bar{\psi}_{-n} \gamma^1 \psi_n \right\} \Big|_{\tau=0}. \quad (29)$$

This boundary effective action is non-local and quadratic in the Fermi field  $\psi$ . Note that we would encounter a divergence in the effective action in the form of product of two delta functions  $(\delta(\tau))^2 = \delta(\tau)\delta(0)$  if the delta function were not present in front of the kinetic term for the Fermi fields  $d$  and  $d^\dagger$  in Eq.(1).

The boundary effective action  $S_{\text{boundary}}[\psi]$  may be written as

$$S_{\text{boundary}} = t^2 \int \frac{d\sigma}{2\pi} \bar{\psi} \gamma^1 [\partial_\sigma]^{-1} \psi \Big|_{\tau=0}, \quad (30)$$

and it leads us to a formal expression of the boundary state

$$|B\rangle = \exp \left[ t^2 \int \frac{d\sigma}{2\pi} \bar{\psi} \gamma^1 [\partial_\sigma]^{-1} \psi \Big|_{\tau=0} \right] |N\rangle \quad (31)$$

where  $|N\rangle$  is the Neumann boundary state for the Fermi fields  $\psi$ . Correlation functions of the physical operators  $\mathcal{O}_i$ ,  $i = 1, \dots, n$  may be evaluated as

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \langle 0 | : \mathcal{O}_1 \cdots \mathcal{O}_n : |B\rangle / \langle 0 | B \rangle. \quad (32)$$

### 3.2 The Exact Boundary State

The explicit expression of the effective boundary action and the boundary state  $|B\rangle$  can be obtained if the two spinor components of the Fermi fields,  $\psi_L$  and  $\psi_R$  are expanded in the Fermion oscillators

$$\psi_L(\tau + i\sigma) = \sum_n \psi_n e^{-n(\tau + i\sigma)} \quad , \quad \psi_L^\dagger(\tau + i\sigma) = \sum_n \psi_n^\dagger e^{-n(\tau + i\sigma)} \quad (33a)$$

$$\psi_R(\tau - i\sigma) = \sum_n \tilde{\psi}_n e^{-n(\tau - i\sigma)} \quad , \quad \psi_R^\dagger(\tau - i\sigma) = \sum_n \tilde{\psi}_n^\dagger e^{-n(\tau - i\sigma)}. \quad (33b)$$

The non-vanishing anticommutation relations between the Fermion operators are

$$\{\psi_m, \psi_n^\dagger\} = \delta_{m+n} \quad , \quad \{\tilde{\psi}_m, \tilde{\psi}_n^\dagger\} = \delta_{m+n}. \quad (34)$$

Since the Fermi fields  $\psi_{L/R}$  are anti-periodic in the NS sector on a cylinder the oscillators  $\psi_n$  are labeled by half-odd-integers,  $n \in \mathbf{Z} + 1/2$ .

The oscillators can be uniquely separated into those with positive and negative modes. The vacuum in the NS sector is unique in contrast to the vacuum in the Ramond sector, which possesses Fermion zero mode oscillators. A representation of (34) is found by beginning with the vacuum state which is annihilated by all positively moded oscillators

$$\left. \begin{aligned} \psi_n |0\rangle_{NS} &= 0, & \tilde{\psi}_n |0\rangle_{NS} &= 0 \\ \psi_n^\dagger |0\rangle_{NS} &= 0, & \tilde{\psi}_n^\dagger |0\rangle_{NS} &= 0 \end{aligned} \right\} \quad n > 0. \quad (35)$$

The Neumann state satisfies [15]

$$\psi_L(0, \sigma) |N\rangle = i\psi_R^\dagger(0, \sigma) |N\rangle, \quad (36a)$$

$$\psi_L^\dagger(0, \sigma) |N\rangle = i\psi_R(0, \sigma) |N\rangle, \quad (36b)$$

which are read in terms of the normal oscillators as

$$\psi_n |N\rangle = i\tilde{\psi}_{-n}^\dagger |N\rangle, \quad \psi_n^\dagger |N\rangle = i\tilde{\psi}_{-n} |N\rangle, \quad (37a)$$

$$\tilde{\psi}_n |N\rangle = -i\psi_{-n}^\dagger |N\rangle, \quad \tilde{\psi}_n^\dagger |N\rangle = -i\psi_{-n} |N\rangle \quad (37b)$$

where  $n$  is a positive half-odd-integer;  $n \in \mathbf{Z} + 1/2$ ,  $n > 0$ . A solution of these equations in the NS sector is

$$|N\rangle =: \exp \left\{ \sum_{n=1/2}^{\infty} i \left( \psi_{-n}^\dagger \tilde{\psi}_{-n}^\dagger + \psi_{-n} \tilde{\psi}_{-n} \right) \right\} : |0\rangle, \quad (38)$$

$$n \in \mathbf{Z} + 1/2, \quad n > 0.$$



An explicit expression of the boundary state  $|B\rangle$  follows from Eqs.(31,33a, 33b)

$$\begin{aligned}
|B\rangle &= : \exp \left\{ -it^2 \int \frac{d\sigma}{2\pi} \left[ \tilde{\psi}^\dagger [\partial_\sigma]^{-1} \tilde{\psi} - \psi^\dagger [\partial_\sigma]^{-1} \psi \right] \right\} : |N\rangle \\
&= : \exp \left\{ -t^2 \sum_n \frac{1}{n} \left( \tilde{\psi}_{-n}^\dagger \tilde{\psi}_n + \psi_{-n}^\dagger \psi_n \right) \right\} : |N\rangle \\
&= \prod_{n=1/2} : \exp \left\{ -\frac{t^2}{n} \tilde{\psi}_{-n}^\dagger \tilde{\psi}_n \right\} :: \exp \left\{ \frac{t^2}{n} \tilde{\psi}_n^\dagger \tilde{\psi}_{-n} \right\} : \\
&\quad : \exp \left\{ -\frac{t^2}{n} \psi_{-n}^\dagger \psi_n \right\} :: \exp \left\{ \frac{t^2}{n} \psi_n^\dagger \psi_{-n} \right\} : |N\rangle.
\end{aligned} \tag{39}$$

We must check whether the constructed boundary state Eq.(39) correctly reproduces the continuity condition Eq.(7) of the model in the unfolded setup. It can be accomplished by rewriting the boundary state  $|B\rangle$  as

$$\begin{aligned}
|B\rangle &= \prod_{n=1/2} : \exp \left\{ -\frac{t^2}{n} \psi_{-n}^\dagger \psi_n \right\} \exp \left\{ -\frac{t^2}{n} \tilde{\psi}_{-n}^\dagger \tilde{\psi}_n \right\} : \\
&\quad \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n}^\dagger \tilde{\psi}_{-n}^\dagger \right\} \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n} \tilde{\psi}_{-n} \right\} |0\rangle.
\end{aligned} \tag{40}$$

Here we make use of the Neumann boundary condition Eqs.(37a,37b). If we apply  $\psi_n$ ,  $n > 0$  on the boundary state  $|B\rangle$  Eq.(40), we find

$$\psi_n |B\rangle = -\frac{t^2}{n} \psi_n |B\rangle + i \left( 1 - \frac{t^2}{n} \right) \tilde{\psi}_{-n}^\dagger |B\rangle. \tag{41}$$

Rearranging this equation we confirm that the obtained boundary condition coincides with the continuity condition in the unfolded setup

$$\psi_n |B\rangle = i \left( \frac{1 - \frac{t^2}{n}}{1 + \frac{t^2}{n}} \right) \tilde{\psi}_{-n}^\dagger |B\rangle. \tag{42}$$

The phase factor  $i$  on the RHS of Eq.(42) is due the phase shift Eq.(13) of the Fermi fields.

We can also check if other boundary conditions are correctly reproduced by applying oscillator operators  $\psi_n^\dagger$ ,  $\tilde{\psi}_n$ ,  $\tilde{\psi}_n^\dagger$ ,  $n > 0$  on the boundary state  $|B\rangle$ . Rewriting the boundary state  $|B\rangle$  as follows, using the Neumann condition Eqs.(24a ,24b) again,

$$\begin{aligned}
|B\rangle &= \prod_{n=1/2} : \exp \left\{ -\frac{t^2}{n} \tilde{\psi}_{-n}^\dagger \tilde{\psi}_n \right\} :: \exp \left\{ -\frac{t^2}{n} \tilde{\psi}_{-n} \tilde{\psi}_n^\dagger \right\} : \\
&\quad : \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n}^\dagger \tilde{\psi}_{-n}^\dagger \right\} :: \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n} \tilde{\psi}_{-n} \right\} : |0\rangle
\end{aligned} \tag{43}$$

we find

$$\tilde{\psi}_n|B\rangle = -\frac{t^2}{n}\tilde{\psi}_n|B\rangle - i\left(1 - \frac{t^2}{n}\right)\psi_{-n}^\dagger|B\rangle. \quad (44)$$

This condition is equivalent to

$$\tilde{\psi}_n|B\rangle = -i\left(\frac{1 - \frac{t^2}{n}}{1 + \frac{t^2}{n}}\right)\psi_{-n}^\dagger|B\rangle. \quad (45)$$

Similarly, if we recast the boundary state into the following form

$$\begin{aligned} |B\rangle &= \prod_{n=1/2} : \exp\left\{-\frac{t^2}{n}\tilde{\psi}_{-n}^\dagger\tilde{\psi}_n\right\} :: \exp\left\{-\frac{t^2}{n}\psi_{-n}\psi_n^\dagger\right\} : \\ &: \exp\left\{i\left(1 - \frac{t^2}{n}\right)\psi_{-n}\tilde{\psi}_{-n}\right\} :: \exp\left\{i\left(1 - \frac{t^2}{n}\right)\psi_{-n}^\dagger\tilde{\psi}_{-n}^\dagger\right\} : |0\rangle, \end{aligned} \quad (46)$$

we find that the boundary state satisfies the following condition

$$\psi_n^\dagger|B\rangle = -\frac{t^2}{n}\psi_n^\dagger|B\rangle + i\left(1 - \frac{t^2}{n}\right)\tilde{\psi}_{-n}|B\rangle, \quad (47)$$

which is equivalent to

$$\psi_n^\dagger|B\rangle = i\left(\frac{1 - \frac{t^2}{n}}{1 + \frac{t^2}{n}}\right)\tilde{\psi}_{-n}|B\rangle. \quad (48)$$

Finally we may rewrite the boundary state  $|B\rangle$  as follows

$$\begin{aligned} |B\rangle &= \prod_{n=1/2} : \exp\left\{-\frac{t^2}{n}\tilde{\psi}_{-n}\tilde{\psi}_n^\dagger\right\} :: \exp\left\{-\frac{t^2}{n}\tilde{\psi}_{-n}^\dagger\tilde{\psi}_n\right\} : \\ &: \exp\left\{i\left(1 - \frac{t^2}{n}\right)\psi_{-n}\tilde{\psi}_{-n}\right\} :: \exp\left\{i\left(1 - \frac{t^2}{n}\right)\psi_{-n}^\dagger\tilde{\psi}_{-n}^\dagger\right\} : |0\rangle. \end{aligned} \quad (49)$$

If we apply  $\tilde{\psi}_n^\dagger$  on the boundary state, we get

$$\tilde{\psi}_n^\dagger|B\rangle = -\frac{t^2}{n}\tilde{\psi}_n^\dagger|B\rangle - i\left(1 - \frac{t^2}{n}\right)\psi_{-n}|B\rangle. \quad (50)$$

This condition can be rearranged as

$$\tilde{\psi}_n^\dagger|B\rangle = -i\left(\frac{1 - \frac{t^2}{n}}{1 + \frac{t^2}{n}}\right)\psi_{-n}|B\rangle. \quad (51)$$

By some explicit algebra, we confirmed that the constructed boundary state in the folded setup correctly reproduces the continuity condition in the unfolded setup.

As  $t$  increases from zero to infinity, the conditions Eqs.(42,45,48,51) interpolate from the Neumann boundary condition Eqs.(37a,37b) to the Dirichlet boundary condition

$$\psi_n|D\rangle = -i\tilde{\psi}_{-n}^\dagger|D\rangle, \quad \psi_n^\dagger|D\rangle = -i\tilde{\psi}_{-n}|D\rangle, \quad (52a)$$

$$\tilde{\psi}_n|D\rangle = i\psi_{-n}^\dagger|D\rangle, \quad \tilde{\psi}_n^\dagger|D\rangle = i\psi_{-n}|D\rangle. \quad (52b)$$

## 4 Correlation Functions of Current Operators

Once we construct the boundary state it becomes a simple matter to calculate the correlation functions of the operators,  $\mathcal{O}_i$ ,  $i = 1, \dots, n$ ,

$$\begin{aligned}\langle T\mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \langle 0| : \mathcal{O}_1 \cdots \mathcal{O}_n : |B\rangle / \langle 0|B\rangle \\ &= \langle 0| : \mathcal{O}_1 \cdots \mathcal{O}_n : \exp(-S_{\text{boundary}}) |N\rangle / \langle 0|B\rangle.\end{aligned}\quad (53)$$

where  $T$  denotes the  $\tau$  ordered product and  $\langle 0|$  is the vacuum state in the NS sector. generacy. So we should define the correlation functions of the operators as follows

As an example, we will calculate the correlation functions of the current operators

$$J_L = \psi_L^\dagger \psi_L = \frac{1}{2}(J^0 - iJ^1), \quad J_R = \psi_R^\dagger \psi_R = \frac{1}{2}(J^0 + iJ^1), \quad (54)$$

where  $J^\mu = \bar{\psi}\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\psi$ . Suppose that a point particle moves to the left (right) toward the contact point. Then after the contact tunneling interaction, it moves back to the right (left). This scattering process may be depicted by the correlation function of  $\langle J_L(\tau, \sigma)J_R(\tau', \sigma') \rangle$  or  $\langle J_R(\tau, \sigma)J_L(\tau', \sigma') \rangle$ . (See Fig. 1.) We expect that the correlation functions  $\langle J_L(\tau, \sigma)J_L(\tau', \sigma') \rangle$  and  $\langle J_R(\tau, \sigma)J_R(\tau', \sigma') \rangle$  are not affected by the contact tunneling interaction. Let us calculate those correlation functions of the current operators.

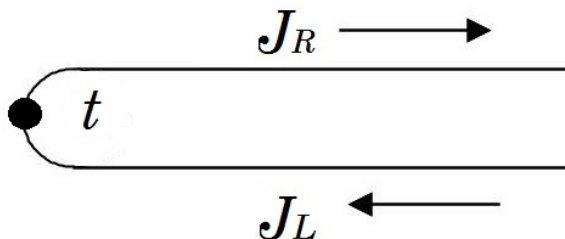


Figure 1: Current Operators on the Half Line

The current operators are written in terms of the oscillator operators as

$$J_L(\sigma) = \sum_{n,m} \psi_n^\dagger \psi_m e^{-i(n+m)\sigma}, \quad J_R(\sigma) = \sum_{n,m} \tilde{\psi}_n^\dagger \tilde{\psi}_m e^{i(n+m)\sigma}, \quad (55)$$

where  $n, m \in \mathbf{Z} + 1/2$ . The correlation function of the two current operators  $J_L$  on the

boundary is given by

$$\begin{aligned}
\langle 0|J_L(\sigma)J_L(\sigma')|B\rangle &= \sum_{p,q,r,s \in \mathbf{Z}+1/2} \langle 0| : \psi_p^\dagger \psi_q :: \psi_r^\dagger \psi_s : \prod_{n=1/2}^{\infty} : \exp \left\{ -\frac{t^2}{n} \psi_{-n}^\dagger \psi_n \right\} \\
&\quad \exp \left\{ -\frac{t^2}{n} \tilde{\psi}_{-n} \tilde{\psi}_n^\dagger \right\} : \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n}^\dagger \tilde{\psi}_{-n}^\dagger \right\} \\
&\quad \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n} \tilde{\psi}_{-n} \right\} |0\rangle e^{-i(p+q)\sigma} e^{-i(r+s)\sigma'} \\
&= \sum_{p,q,r,s \in \mathbf{Z}+1/2} \langle 0| : \psi_p^\dagger \psi_q :: \psi_r^\dagger \psi_s : |0\rangle e^{-i(p+q)\sigma} e^{-i(r+s)\sigma'}. \tag{56}
\end{aligned}$$

The vacuum on the LHS is annihilated unless  $p$  and  $q$  are positive half-odd integers. Since the current operators do not have the right movers, we should drop the terms containing the right movers. The only non-vanishing components are those with  $r = -p$ ,  $s = -q$ . The current correlation function  $\langle J_L J_L \rangle$  is not affected by the point contact interaction as expected. Thus,

$$\begin{aligned}
\langle 0|J_L(\sigma)J_L(\sigma')|B\rangle &= \sum_{p,q=1/2}^{\infty} e^{-ip(\sigma-\sigma')} e^{-iq(\sigma-\sigma')} \\
&= \frac{zz'}{(z-z')^2}. \tag{57}
\end{aligned}$$

where  $z = e^{i\sigma}$ ,  $z' = e^{i\sigma'}$ . The correlation function of the current operators  $\langle 0|J_L(\tau, \sigma)J_L(\tau', \sigma')|B\rangle$  is obtained from Eq.(57)  $\langle 0|J_L(\sigma)J_L(\sigma')|B\rangle$  simply by replacing  $z$  and  $z'$  as

$$z = e^{i\sigma} \rightarrow z = e^{\tau+i\sigma}, \quad z' = e^{i\sigma'} \rightarrow z' = e^{\tau'+i\sigma'}. \tag{58}$$

A similar procedure works also for  $\langle 0|J_R(\sigma)J_R(\sigma')|B\rangle$

$$\begin{aligned}
\langle 0|J_R(\sigma)J_R(\sigma')|B\rangle &= \sum_{p,q,r,s} \langle 0| : \tilde{\psi}_p^\dagger \tilde{\psi}_q :: \tilde{\psi}_r^\dagger \tilde{\psi}_s : \prod_{n=1/2}^{\infty} : \exp \left\{ -\frac{t^2}{n} \psi_{-n}^\dagger \psi_n \right\} \\
&\quad \exp \left\{ -\frac{t^2}{n} \tilde{\psi}_{-n} \tilde{\psi}_n^\dagger \right\} : \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n}^\dagger \tilde{\psi}_{-n}^\dagger \right\} \\
&\quad \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n} \tilde{\psi}_{-n} \right\} |0\rangle e^{i(p+q)\sigma} e^{i(r+s)\sigma'} \\
&= \sum_{p,q,r,s} \langle 0| : \tilde{\psi}_p^\dagger \tilde{\psi}_q :: \tilde{\psi}_r^\dagger \tilde{\psi}_s : |0\rangle e^{i(p+q)\sigma} e^{i(r+s)\sigma'} \\
&= \sum_{p=1/2, q=1/2}^{\infty} e^{ip(\sigma-\sigma')} e^{iq(\sigma-\sigma')} \\
&= \frac{\bar{z}\bar{z}'}{(\bar{z}-\bar{z}')^2} \tag{59}
\end{aligned}$$

where  $\bar{z} = e^{-i\sigma}$ ,  $\bar{z}' = e^{-i\sigma'}$ . The current correlation function  $\langle J_R J_R \rangle$  is also unaffected by the point contact interaction as expected.

The only non-trivial correlation function is  $\langle J_L J_R \rangle$ , which is given by

$$\begin{aligned}
\langle 0 | J_L(\sigma) J_R(\sigma') | B \rangle &= \sum_{p,q,r,s} \langle 0 | : \psi_p^\dagger \psi_q : : \tilde{\psi}_r^\dagger \tilde{\psi}_s : | B \rangle e^{-i(p+q)\sigma} e^{i(r+s)\sigma'} \\
&= \sum_{p,q,r,s} \langle 0 | : \psi_p^\dagger \psi_q : : \tilde{\psi}_r^\dagger \tilde{\psi}_s : \prod_{n=1/2}^{\infty} : \exp \left\{ -\frac{t^2}{n} \psi_{-n}^\dagger \psi_n \right\} \\
&\quad \exp \left\{ -\frac{t^2}{n} \tilde{\psi}_{-n}^\dagger \tilde{\psi}_n \right\} : \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n}^\dagger \tilde{\psi}_{-n} \right\} \\
&\quad \exp \left\{ i \left( 1 - \frac{t^2}{n} \right) \psi_{-n} \tilde{\psi}_{-n} \right\} | 0 \rangle e^{i(p+q)\sigma} e^{i(r+s)\sigma'}. \tag{60}
\end{aligned}$$

By some algebra, we find

$$\begin{aligned}
\langle 0 | J_L(\sigma) J_R(\sigma') | B \rangle &= \sum_{p,q=1/2}^{\infty} \left( 1 - \frac{t^2}{p} \right) \left( 1 - \frac{t^2}{q} \right) e^{-i(p+q)(\sigma-\sigma')} \\
&= \left( \sum_{p=1/2}^{\infty} \left( 1 - \frac{t^2}{p} \right) e^{-ip(\sigma-\sigma')} \right)^2. \tag{61}
\end{aligned}$$

We may define the mode (frequency) dependent conductance  $G_{pq}$  as

$$\langle 0 | J_L(\tau, \sigma) J_R(\tau', \sigma') | B \rangle = \sum_{p,q=1/2}^{\infty} (G_{pq} + 1) (z \bar{z}')^{-p} (z \bar{z}')^{-q}, \tag{62a}$$

$$G_{pq} = \left( 1 - \frac{t^2}{p} \right) \left( 1 - \frac{t^2}{q} \right) - 1 \tag{62b}$$

where  $z = e^{\tau+i\sigma}$  and  $\bar{z}' = e^{\tau'-i\sigma'}$ . The frequency dependency of the conductance is due to the non-local nature of the contact tunneling interaction. The contact interaction acts on the lower frequency modes more strongly. The high frequency modes are affected only weakly by the contact tunneling interaction. For small  $t$ ,  $G_{pq} < 0$ . Eq.(62b) indicates that the interaction suppresses the conductance and its effect decreases as the frequency increases. Using the following series expansion

$$\ln \left( \frac{1+x}{1-x} \right) = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \right), \tag{63}$$

we easily obtain the current correlation function  $\langle 0 | J_L(\tau, \sigma) J_R(\tau', \sigma') | B \rangle$  in a closed form

$$\langle 0 | J_L(\tau, \sigma) J_R(\tau', \sigma') | B \rangle = \left( \frac{\sqrt{z \bar{z}'}}{z \bar{z}' - 1} - 2t^2 \ln \left( \frac{\sqrt{z \bar{z}'} + 1}{\sqrt{z \bar{z}'} - 1} \right) \right)^2, \tag{64}$$

where  $z = e^{\tau+i\sigma}$  and  $\bar{z}' = e^{\tau'-i\sigma'}$ .

The correlation function as given by Eq.(64) is valid only when  $t$  is small, since we define the boundary state  $|B\rangle$  as a perturbative state from the Neumann state, corresponding to the state with  $t = 0$ . As  $t$  increases the point-contact tunneling interaction becomes strong. But for a large value of  $t$  we cannot simply extrapolate the correlation function Eq.(61) and Eq.(64) which are valid only for small  $t$ . In the large  $t$  limit, the boundary state turns into the Dirichlet state Eq.(52a,52b). In this strong coupling region, we need to redefine the perturbation theory, around the Dirichlet state. It may be accomplished by the T-dual transformation of string theory.

## 5 Conclusions

In this paper we discuss the model of resonant point-contact tunneling of a single Luttinger-liquid lead, which is the simplest model of this type. The model is initially defined on an infinite line with a single chiral Fermion and a mid-point contact interaction. In order to apply the boundary state formulation of string theory, we recast the model in the unfolded setup into the folded setup. In the folded setup the model is described by a Dirac spinor with two components. The Fermi field operators, corresponding to the creation and annihilation operators of charge on the resonant state, are also extended to a spinor with two components. Integrating out this Fermi field yields the boundary effective action, which is nonlocal. Since the effective boundary action is only quadratic in the field operator, the boundary state can be explicitly constructed. The constructed boundary state is shown to reproduce correctly the continuity condition of the model in the unfolded setup as boundary condition.

Correlation functions of the physical operators can be calculated by using the explicit expression of the boundary state. As an immediate application the correlation functions of current operators are evaluated. Despite the simple structure of the model, it shares the intriguing features of the fully fledged models. The point contact interaction acts on the lower frequency modes more strongly. The high frequency modes are only weakly affected by the interaction. The folding procedure and the boundary state formulation are certainly applicable to more complex models, which include the resonant multilead point-contact tunneling [1], the quantum Brownian motion on a honeycomb lattice [1, 21, 22]. Extensions of this work along this direction will be given somewhere else.

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